GRAPHICAL REPRESENTATION OF FUNCTIONS OF THE *N*TH DEGREE.

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(*Read April 26, 1919.*)

It has been generally assumed that any algebraic symbol y, the exponent of which is unity, can only be properly represented graphically by a line. When the exponent is 2, the quantity y^2 has been assumed to represent a rectangular area, the four sides of which each have a length y. If the exponent is 3, the proper geometrical representation was said to be a rectangular volume, having six faces whose areas were y^2. If the exponent were 4, the philosophic mind began to speculate on the nature of space having four dimensions. No definite solution of this problem having been reached, it has apparently been considered impracticable to consider the properties of space having six or eight or ten dimensions.

During this period of uncertainty it has been known to students of elementary algebra that

$$\frac{1}{y^{1/3}} = \frac{y^{1/3}}{y^{2/3}} = \frac{y^{2/3}}{y} = \frac{y}{y^{4/3}} = \frac{y^{4/3}}{y^{5/3}} = \frac{y^{5/3}}{y^2}, \quad \text{etc.}$$

It has also been known that similar right triangles have sides whose lengths are directly proportional. This is illustrated in Fig. 1 where an arbitrarily chosen length is adopted as a unit of length. Its length in terms of the centimeter is unknown. Another length is similarly chosen to represent $y^{\frac{1}{3}}$, where y is to be given a numerical value in terms of the chosen unit, but this value need not be determined.

These lengths having been laid off upon rectangular axes, the lengths $y^{2/3}$, y, $y^{4/3}$, $y^{5/3}$, y^2, etc., can be laid off upon the two axes by a well-known geometrical method. By means of three parallel rulers, two of which are linked to the third in the usual manner, the lengths unity and y may be first adopted and laid off on both axes, and the lengths $y^{\frac{2}{3}}$ and $y^{\frac{1}{3}}$ can then be laid off. By means of two rulers in parallel, the length $y^{\frac{1}{2}}$ can similarly be laid off upon both axes. The lengths $y^{3/2}$, $y^{5/2}$, etc., can then be similarly laid off upon the two axes. Since

$$\frac{y}{1}=\frac{xy}{x}=\frac{x^2y}{x^2}=\frac{x^3y}{x^3}, \quad \text{etc.,}$$

it is evident that any value x^ny^m can be laid off by purely geometrical methods. The exponents n and m may be in decimal form and may range in numerical value from 0.1 to 100 if the experimenter so desires, and all of these values may be represented by lines.

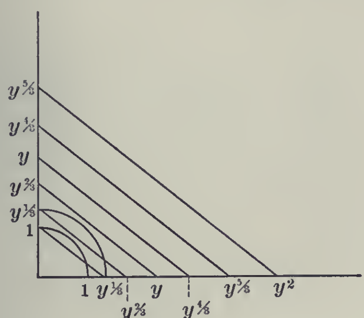


FIG. 1.

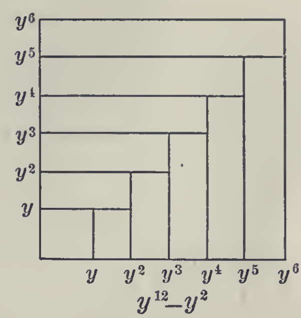


FIG. 2.

In Fig. 2 we have a system of squares constructed on lines determined by the method described above. The areas of the squares vary from y^2 to y^{12}. The difference between the areas of the outer and the inner squares is $y^{12}-y^2$.

This value is divisible by y^2-y. The value so determined is

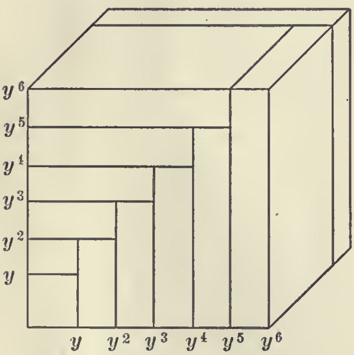
$$\frac{y^{12}-y^2}{y^2-y}=y^{10}+y^9+y^8+y^7+y^6+y^5+y^4+y^3+y^2+y.$$

Multiplying both members of this equation by y^2-y, it may be put into the form

$$\begin{aligned}y^{12}-y^2=y^6(y^6-y^5)&+y^5(y^6-y^5)\\+y^5(y^5-y^4)&+y^4(y^5-y^4)\\+y^4(y^4-y^3)&+y^3(y^4-y^3)\end{aligned}$$

$$+ y^3(y^3 - y^2) + y^2(y^3 - y^2)$$
$$+ y^2(y^2 - y\ ) + y\ (y^2 - y\ ). \qquad (1)$$
$$y^{12} - y^2.$$

The first two terms of the second member of this equation represent the area of the two outer rectangles between the two squares having areas y^{12} and y^{10}. These rectangles each have a width $y^6 - y^5$, and their combined length is $y^6 + y^5$. The remaining terms represent the remaining strip areas, between the squares y^{12} and y^2,



$y^{18} - y^3$

FIG. 3.

$$y^{18} - y^3.$$

Fig. 3 represents a cube, the volume within the outer surface of which is y^{18}. In the lower left-hand corner is a cube having a volume y^3. Between these cubes are fifteen blocks filling the space between the two cubes. If this cube were to be placed upon the area forming Fig. 2, each block which stands on edge would cover a rectangular area shown in Fig. 2. Dividing the difference between the volumes by $y^2 - y$ as in the former case we have

$$\frac{y^{18} - y^3}{y^2 - y} = y^{16} + y^{15} + \cdots y^2.$$

Multiplying by $y^2 - y$ as before the resulting terms may be written

$$y^{18} - y^3 = y^6y^6(y^6 - y^5) + y^6y^5(y^6 - y^5) + y^5y^5(y^6 - y^5)$$
$$+ y^5y^5(y^5 - y^4) + y^5y^4(y^5 - y^4) + y^4y^4(y^5 - y^4)$$
$$+ y^4y^4(y^4 - y^3) + y^4y^3(y^4 - y^3) + y^3y^3(y^4 - y^3)$$
$$+ y^3y^3(y^3 - y^2) + y^3y^2(y^3 - y^2) + y^2y^2(y^3 - y^2)$$
$$+ y^2y^2(y^2 - y\ ) + y^2y\ (y^2 - y\ ) + y\ y\ (y^2 - y\ ). \qquad (2)$$

The first term in the second member of the equation represents the volume of the block at the rear of the cube y^{18}. Its dimensions are y^6, y^6 and $y^6 - y^5$. The second term represents the volume of the block on the right side of the cube y^{18}. The dimensions are y^6, y^5 and $y^6 - y^5$. The third term gives the volume of the block covering the upper face of the cube y^{15}. The volume of the three blocks is $y^{18} - y^{15}$. This will be the result obtained by adding these three terms together. Each of these three terms may be given a very different interpretation. They are each a difference between two quantities. They are therefore each the difference between the cubes of two other quantities,

$$y^6 y^6 (y^6 - y^5) = y^{18} - y^{17}.$$

This result shows that the volume of the block forming the back of the cube whose volume is y^{18} is also equal to the volume of three blocks between the cubes y^{18} and y^{17}. We may therefore write the value of this term as follows:

$$\begin{aligned} y^{18} - y^{17} = {} & y^6\ y^6\ (y^6 - y^{17/3}) \\ & + y^6\ y^{17/3}(y^6 - y^{17/3}) \\ & + y^{17/3} y^{17/3}(y^6 - y^{17/3}). \end{aligned}$$

In this case the inner cube would have edges whose length is $y^{17/3}$. This length can be laid off on the three axes by purely geometrical means, as has been pointed out in the present paper.

The second term of Eq. (2), which represents the volume of the block on the right side of Fig. 2, may also represent the volume of three blocks between cubes y^{17} and y^{16}. The third term representing the block at the top of the large cube may also represent the volume $y^{16} - y^{15}$. These nine blocks would fill the volume occupied by the three outer blocks of Fig. 3.

The thickness of the three shells filling the volume $y^{18} - y^{15}$ is $(y^6 - y^{17/3}) + (y^{17/3} - y^{16/3}) + (y^{16/3} - y^5)$. Of course, all of the terms of Eq. (2) can be similarly treated, and the operation may be repeated on the terms resulting.

Equation (2) may also represent the difference between the areas of two squares, y^{18} and y^3, having sides whose lengths are y^9 and $y^{3/2}$. As the second member has an odd number of terms, the final term is resolved into two terms by considering it to be the difference between two squares having areas y^4 and y^3. The equation may therefore be written

$$\begin{aligned} y^{18} - y^3 = y^9(y^9 - y^8) &+ y^8(y^9 - y^8) \\ + y^8(y^8 - y^7) &+ y^7(y^8 - y^7) \\ + y^7(y^7 - y^6) &+ y^6(y^7 - y^6) \\ + y^6(y^6 - y^5) &+ y^5(y^6 - y^5) \\ + y^5(y^5 - y^4) &+ y^4(y^5 - y^4) \\ + y^4(y^4 - y^3) &+ y^3(y^4 - y^3) \\ + y^3(y^3 - y^2) &+ y^2(y^3 - y^2) \\ + y^2(y^2 - y^{3/2}) &+ y^{3/2}(y^2 - y^{3/2}). \end{aligned}$$

Equation (1) may also represent the difference between the volumes y^{12} and y^2 of two cubes having edges whose lengths are y^4 and $y^{2/3}$. That equation may be written

$$\begin{aligned} y^{12} - y^2 = y^4y^4(y^4 - y^3) &+ y^4y^3(y^4 - y^3) + y^3y^3(y^4 - y^3) \\ + y^3y^3(y^3 - y^2) &+ y^3y^2(y^3 - y^2) + y^2y^2(y^3 - y^2) \\ + y^2y^2(y^2 - y\ ) &+ y^2y\ (y^2 - y\ ) + y\ y\ (y^2 - y\ ) \\ + y\ y\ (y - y^{2/3}) &+ y\ y^{2/3}(y - y^{2/3}) + y^{2/3}y^{2/3}(y - y^{2/3}). \end{aligned}$$

Here the final term $(y^3 - y^2)$ of Eq. (1) has been considered as the difference between the volumes of two cubes having edges the lengths of which are y and $y^{2/3}$.

The last equation may be written

$$y^{12} - y^2 = [(y^4)^3 - (y^3)^3] + [(y^3)^3 - (y^2)^3] + [(y^2)^3 - y^3] + [y^3 - y^2].$$

Of course, this discussion does not in any way modify the physical or geometrical meaning of the units linear foot, square foot or cubic foot. A contractor who should order of dealers in such material 64 feet of garden-hose, 64 feet of sheet tin and 64 feet of sand or gravel, would not learn that any of his orders had been misunderstood. Possibly they might not be understood if he were to order 2^6 feet of each of these materials.